TORIC GEOMETRY

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Abstract

In this first session, we familiarize ourselves with the combinatorics of fans and describe the relationship with toric varieties. We provide many examples

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0 Overview

1 Toric Varieties

Definition 1.0.1. A toric variety over $\mathbb{C}$ is an (irreducble) abstract variety X with an embedding $(\mathbb{C}^*)^d \hookrightarrow X$ such that $(\mathbb{C}^*)^d$ is Zariski-open and such that the canonical multiplication on $(\mathbb{C}^*)^d$ extends to an action of $(\mathbb{C}^*)^d$ on X

Here are some examples:

Example 1.0.2.

- *Obviously* $(\mathbb{C}^*)^n$ *and* $\mathbb{A}^n$.
- $\mathbb{P}^n$ *is toric as follows: consider the map*

$$(\mathbb{C}^*)^n \longrightarrow U_0 \stackrel{\text{def}}{=} \mathbb{P}^n \setminus \{x_0 = 0\} : (t_1, \ldots, t_n) \mapsto (1, t_1, \ldots t_n)$$

This identifies $(\mathbb{C}^*)^n$ *with the open subvariety* $\mathbb{P}^n \setminus \mathbb{V}(x_0 \cdot \ldots \cdot x_n)$ *and the rule*

$$(t_1, \ldots, t_n) \cdot (a_0, \ldots, a_n) \stackrel{\text{def}}{=} (a_0, t_1 a_1, \ldots t_n a_n)$$

extends the action. (note that this choice is not canonical)

- *There is a technique which provides a toric structure on affine varieties which have a specific type of parametrization. Assume X is an affine variety equipped with a bijective morphism*

$$\Phi : \mathbb{C}^n \xrightarrow{\sim} X$$

Then the restriction defines an embedding $(\mathbb{C}^*)^n \hookrightarrow X$ as a Zariski open subset. Moreover if the component functions $\Phi \stackrel{\text{def}}{=} (\phi_1, \ldots, \phi_d)$ have the property that they are multiplicative in the sence that

$$(\phi_1(\underline{s}), \ldots \phi_d(\underline{s})) \cdot (\phi_1(\underline{t}), \ldots, \phi_d(\underline{t})) = (\phi_1(\underline{st}), \ldots \phi_d(\underline{st}))$$

Then it is easy to see that the action

$$\underline{t} \cdot (x_1, \ldots x_d) \stackrel{\text{def}}{=} (\phi_1((\underline{t})x_1, \ldots \phi(\underline{t})x_n)$$

will preserve any polynomial relation satisfied by $(\phi_1(\underline{s}), \ldots \phi_d(\underline{s}))$ and hence lift the action of $(\mathbb{C}^*)^n$ on itself. Below we list some specific examples:

- *the cuspidal cubic $X \stackrel{\text{def}}{=} \mathbb{V}(y^2 - x^3)$ has a paramametrization given by (t^2, t^3), as a result we have the embedding $\mathbb{C}^* \longrightarrow X : t \mapsto (t^2, t^3)$. $\mathbb{C}^*$ acts on X by $t \cdot (u, v) \stackrel{\text{def}}{=} (t^2 u, t^3 v)$ since explicitely*

$$(t^2 u)^3 - (t^3 v)^2 = t^6 u^3 - t^6 v^2 = t^6(u^3 - v^2) = 0$$

Note that the parametrization is multiplicative

- *the conifold $X = \mathbb{V}(xy - zw)$ in $\mathbb{A}^4$ has a parametrization*

$$(t_1, t_2, t_3) \longrightarrow (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$$

and hence a canonical embedding of $(\mathbb{C}^*)^3$. The action is extended on X by

$$(t_1, t_2, t_3) \cdot (x, y, z, w) \stackrel{\text{def}}{=} (t_1 x, t_2 y, t_3 z, t_1 t_2 t_3^{-1} w)$$

again as the parametrization is multiplicative

- *the surface $X \stackrel{\text{def}}{=} \mathbb{V}(xz - y^2)$ (the A_1-Du Val singularity, or the cone over the parabola $\mathbb{V}(x - y^2)$) has the parametrization*

$$(\mathbb{C}^*)^2 \longrightarrow X : (t_1, t_2) \mapsto (t_1^2, t_1 t_2, t_2^2)$$

which is again multiplicative and hence X is toric

- *In a different context, the weighted projective line $\mathbb{P}(q_0, \ldots q_n)$ (where $\gcd(q_1, \ldots, q_n) = 1$) which is defined as $\mathbb{C}^{n+1}$ modulo the relation*

$$(a_1, \ldots, a_n) \sim (b_1 \ldots b_n) \iff \exists \lambda \in \mathbb{C}^*, (b_1 \ldots b_n) = (\lambda^{q_1} a_0, \ldots \lambda^{q_n} b_n)$$

is also toric as follows: note that the image of $(\mathbb{C}^*)^{n+1}$ in $\mathbb{P}(q_1, \ldots q_n)$ is $(\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ where $\mathbb{C}^*$ is a subgroup by the map $\lambda \mapsto (\lambda^{q_1}, \ldots \lambda^{q_n})$. Now this group is canonically isomorphic to $(\mathbb{C}^*)^n$ by

$$((\mathbb{C}^*)^{n+1}/\mathbb{C}^* \xrightarrow{\sim} (\mathbb{C}^*)^n : (x_0, \ldots x_n) \mapsto \left(\frac{x_1}{x_0^{q_0}}, \ldots, \frac{x_n}{x_0^{q_n}} \right)$$

In this way we obtain an embedding $(\mathbb{C}^*)^n \hookrightarrow \mathbb{P}(q_1, \ldots q_n)$ and the action of $\mathbb{C}^{n+1}$ induces an action of $(\mathbb{C}^*)^n$ on $\mathbb{P}(q_1, \ldots q_n)$.
If you haven't worked with these weighted projective spaces, it's a good exercise to look at $\mathbb{P}(1, 1, 2)$: as in the usual case we look at subsets

$$U_i \stackrel{\text{def}}{=} \{[a_0, \ldots, a_n] \,|\, a_i \neq 0\}$$

The idea is that the weights now turn these U_i 's in (strict) subvarieties as opposed to affine spaces. since we have

$$\left\{ \begin{array}{ll} \textit{on } U_0 : & [a_0, a_1, a_2] = [1, \frac{a_1}{a_0}, \frac{a_2}{a_0^2}] \\ \textit{on } U_1 : & [a_0, a_1, a_2] = [\frac{a_0}{a_1}, 1, \frac{a_0}{a_1^2}] \end{array} \right\}$$

It 's clear that the first two sets are simply affine space again. this trick doesn 't work for U_2 however. Here instead, invariant theory yields that $U_2 \longrightarrow \mathbb{C}^3 : [a_0, a_1, a_3] \mapsto (\frac{a_0^2}{a_3}, \frac{a_0 a_1}{a_3}, \frac{a_1^2}{a_0})$ is an embedding, yielding that U_2 is isomorphic to the A_1 singularity $\mathbb{V}(xz - y^s)$ we saw above.

1.1. Polyhedral Cones

The main philosophy in toric geometry is that certain combinatorial gadgets can be used to define (affine) toric varieties. The following observation conveys the idea:

Lemma 1.1.1. *There is an isomorphism*

$$\mathbb{Z}^n \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{C}) : (a_1, \ldots a_n) \longrightarrow (x_1, \ldots x_n) \mapsto x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$$

As such the algebra generated by the group of characters corresponds to a torus. The idea is now to take different lattices in $\mathbb{R}^n$ to create toric varieties in a similar way. This turns out to have some technical issues. To resolve this, we change the base ring to $\mathbb{R}$ and keep track of the underlying lattice.

Convention 1.1.2. *Throughout, we fix a lattice M, and denote its dual lattice $N \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}}(M, Z)$ and the canonical pairing will be denoted $\langle -, - \rangle$*

Definition 1.1.3. A polyhedral cone in $\mathbb{R}^n$ is a set of the form

$$\sigma \stackrel{\text{def}}{=} \mathrm{cone}(S) \stackrel{\text{def}}{=} \{\sum_{p \in S} \lambda_p p \,|\, \lambda_p \geq 0\}$$

This set is always

- convex since for x, y, $\lambda x + (1 - \lambda) y \in \sigma$ for $0 \leq \lambda \leq 1$. Indeed we have

$$\lambda\big(\sum_p \alpha_p p\big) + (1 - \lambda)\big(\sum_p \beta_p p\big) = \sum_p \big(\lambda_p \alpha_p + (1 - \lambda_p)\beta_p\big) p$$

  and it's an easy exercise to see that these coefficients are positive if $0 \leq \lambda \ \leq 1$

- conic as $x \in \sigma \implies \lambda x \in \sigma$ for $\lambda \geq 0$

Definition 1.1.4. Let σ be a polyhedral cone. Its dual is given by

$$\sigma^\vee \stackrel{\text{def}}{=} \{u \in (\mathbb{R}^n)^* \,|\, \langle u, v \rangle \geq 0, u \in \sigma\}$$

(i.e. all vectors whose angle $\leq |\frac{\pi}{2}|$) with σ.
If $u \in (\mathbb{R}^n)^*$, we get the hyperplane

$$\mathrm{H}_u \stackrel{\text{def}}{=} (u)^\perp$$

and the half-space

$$\mathrm{H}_u^+ \stackrel{\text{def}}{=} \{v \in \mathbb{R}^n \,|\, \langle v, u \rangle \geq 0\}$$

Similarly, $v \in \mathbb{R}^n$ defines a hyperplane and half space in $(\mathbb{R}^n)^*$

Definition 1.1.5. A *face* is a subset of σ of the form $\mathrm{H}_u \cap \sigma$ for $u \in \sigma^\vee$. A *facet* is a face of codimension 1

It is easy to see that for $\sigma = \mathrm{cone}(S)$, we have $\sigma^\vee = \cap_{u \in S} H_u^+$. This is how one usually draws the dual polyhedral cone.

Lemma 1.1.6. *We have* $\sigma^{\vee\vee} = \sigma$

Proof. We defer the proof till later ✗

the following property will be important for our applications:

Proposition 1.1.7. *Let σ be a polyhedral cone. TFAE:*

- $\sigma \cap -\sigma = 0$
- *sigma doesn't contain a line*
- 0 *is a face*
- $\dim \sigma^\vee = n$

We won't prove this proposition, as it's inuitive enough. Instead we make the following:

Definition 1.1.8. A polyhedral cone which satisfies the above is *strongly convex*

finally, we will be conncered with knowing when the cone comes from a lattice. So pick a cone $\sigma \in N \otimes \mathbb{R}$. We say that it's rational if $\sigma = \mathrm{cone}(S)$ for $(S) \in N$

Intuitively (assuming some choice of basis), a face is made by intersecting the cone with the zero locus of an equation which yields positive values on the cone itself, or put differently which cuts the space in such a way that the cone stays is completely on one side. Here are some properties of faces:

Lemma 1.1.9.

- *a face is a polyhedral cone*
- *an intersection of two faces is a face*
- *a face of a face of σ is a face of σ*

Proof. The first point is trivial: let $\sigma = \mathrm{cone}(S)$ then $\mathrm{H}_u \cap \sigma = \mathrm{cone}(S \cap H_u)$.

For the second point, we must show that $\sigma \cap \mathrm{H}_u \cap \mathrm{H}_{u'}\ \sigma \cap \mathrm{H}_{u+u'}$ Indeed, the left inclusion is trivial. For the right inclusion, assume that $\langle x, u + u' \rangle = 0$,then $\langle x, u \rangle + \langle x, u' \rangle = 0$ but both numbers are positive as $u, u' \in \sigma^\vee$ by definition.

For the third point, first some intuition. We cut a cone σ along a plane which keeps σ to one side to obtain a cone τ and perform the same operation with a plane which now keeps τ to one side and we must show that we can 'incline' this plane so that it actually keeps the whole of σ to one side.
Consider $\tau = \sigma \cap \mathrm{H}_u$ with $u \in \sigma^\vee$ and $\gamma = \tau \cap \mathrm{H}_{u'}$ for $u' \in \tau^\vee$. Assume $\sigma = \mathrm{cone}(S)$ and $\tau = \mathrm{cone}(S')$. Then $\langle S \setminus S', u \rangle \geq 0$ and $\langle S', u' \rangle \geq 0$, since all sets are finite, by picking λ large enough, we can ensure that $\langle S, u' + \lambda u \rangle \geq 0$, i.e. $u' + \lambda u \in \sigma^\vee$. Now, we claim that

$$\tau \cap \mathrm{H}_{u'} = \sigma \cap \mathrm{H}_{u' + \lambda u}$$

Indeed, the lhs is $\sigma \cap \mathrm{H}_u \cap \mathrm{H}_{u'}$, and the left inclusion is trivial. Conversely, picking an element in σ such that $\langle x, u' + \lambda u \rangle = 0$, implies $\langle x, u' \rangle + \lambda \langle x, u \rangle = 0$, now the second number is certainly postive as $u \in \sigma^\vee$, but the first also since otherwise we can find an element $y \in \sigma$ such that $\langle y, u + \lambda u' \rangle < 0$ by considering the component of x which lies on H_u ✗

A little more technical to prove is:

Lemma 1.1.10. *Any face is the intersection of its facets.*

Proof. This reduces to showing that any face of codimension greater than 1 is contained in a larger face. We omit the proof ✗

One can reconstruct the cone and its dual out of its facets, for this, we use an intuitive topological fact:

Lemma 1.1.11. *the boundary of a polyhedral cone is the union of its faces (or facets)*

Theorem 1.1.12. *Assume the facets of* σ *are given by* $\tau_i \overset{\text{def}}{=} \sigma \cap \mathrm{H}_{u_i}$*, then*

- $\sigma = \mathrm{H}^+_{u_1} \cap \dots \mathrm{H}^+_{u_n}$
- $\sigma^\vee = \mathrm{cone}(u_1, \dots u_n)$

Proof. For the first item, the right inclusion is obvious. Conversely, assume v lies in the intersection of these halfplanes but not in σ. Take a point in the interiour of σ and consider the line segment $\overline{vv'}$ this is compact and hence has a furthest point $w \in \sigma$. Then w lies in the boundary of σ and hence on a facet. $\tau \overset{\text{def}}{=} \sigma \cap H_{u_\tau}$. Then $\langle w, u_t a u \rangle = 0$, $\langle v', u_\tau \rangle \geq 0$ and since w is the furthest point, $\langle v, u_\tau \rangle < 0$, a contradiction.
We do not prove the second point. ✗

proof of lemma 1.1.6 . ✗

2 Affine Toric Varieties

We consider a lattice N and let $M \overset{\text{def}}{=} \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$ denote the dual lattice. We say that a polyhedral cone is rational if it is of the form $cone(S)$ for $S \in N$ For a rational polyhedral cone σ, we define:

Definition 2.0.13. $S_\sigma \overset{\text{def}}{=} \sigma^\vee \cap M$

Lemma 2.0.14. S_σ *is a semigroup*

Proof. Trivial ✗

Theorem 2.0.15 (Gordan). S_σ *is finitely generated*

Proof. let $\sigma = \mathrm{cone}(u_1, \dots, u_n)$ and $K \overset{\text{def}}{=} \{\sum t_i u_i \, | 0 \leq t_i \leq 1\}$. Then $K \cap M$ is finite, being an intersection of a compact with a discrete. Then $K \cap M$ is a generator for S_σ. Indeed, let $u = \sum \lambda_i u_i \in \sigma^\vee$, then $\lambda_i = m_i + t_i$ with $0 \leq t_i \leq 1$ and $m_i \in \mathbb{N}$ and $u = \sum m_i u_i + \sum t_i u_i$ and both u_i and $\sum t_i u_i$ lie in $K \cap M$ ✗

We now consider the semigroup algebra $\mathbb{C}[S_\sigma]$ and sometimes denote the basis elements by χ^u for $u \in S_\sigma$ and write $\chi^u \cdot \chi^{u'} = \chi^{u+u'}$

Lemma 2.0.16. $\mathbb{C}[S_\sigma]$ *is an integral domain*

Proof. trivial since S_σ is a lattice ✗

the above lemmas allow us to make the following definition

Definition 2.0.17. Let N be a lattice and σ a scrap. Then the toric variety of σ is defined as

$$X = \mathrm{Spec}(\mathbb{C}[S_\sigma])$$

We shall prove that X_σ is a toric variety through a series of lemmas. The key fact here is how the semigroup of a cone relates to the semigroup of one of its faces:

Lemma 2.0.18. *Let σ be a cone with face $\tau = H_m \cap \sigma$. Then*

$$S_\tau = S_\sigma + \mathbb{N}(-m)$$

Proof. Intuitively, this states that any element w in $\tau^\vee$ may be negative on σ, however, there is a large enough α so that $\mathtt{w} + \alpha u$ is nonnegative on the whole of σ ✗

This allows us to compare the affine varieties associated to cones and faces:

Lemma 2.0.19. *Let σ be a cone with face $\tau = H_u \cap \sigma$. Then*

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi_u}$$

Hence X_τ is a principal open subset of σ

Proof. This follows immediately from the above ✗

Theorem 2.0.20. *for any scrap, X_σ is an affine toric variety.*

Proof. We know that X_σ is an irreducible affine variety. Now σ is *strongly* convex, 0 is a face of σ. Hence we have the open distinguished corresponding $\mathrm{Spec}(\mathbb{C}[S_0])$, which we know is a torus by the first example below. This shows that X_σ contains an open torus. To see how the action of the torus extends to the whole of X_σ, we introduce generators for the semigroup S_σ, $(m_1, \ldots, m_r)$. Then we have a composition

$$\mathbb{C}[x_1, \ldots, x_r] \longrightarrow \mathbb{C}[s_\sigma] \longrightarrow \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

where we identified characters on $\mathbb{Z}^n$ with Laurent polynomials. On the associated varieties, we see that the torus is embedded in S_σ as follows:

$$(\mathbb{C}^*)^r \longrightarrow \mathbb{C}^n : (t_1, \ldots t_n) \mapsto (t_1^{m_1}, \ldots t_r^{m_r})$$

The morphism extends to an action of the torus $(\mathbb{C}^*)^r$ on the whole of $\mathbb{C}^n$ by

$$(t_1 \ldots t_n) \cdot (x_1, \ldots, x_r) \stackrel{\text{def}}{=} (t_1^{m_1} x_1 \ldots t^{m_r} x_r)$$

The density of $(\mathbb{C}^*)^r$ in X_σ implies that this action restricts to one on X_σ ✗

We conclude this section by giving some examples:

Example 2.0.21.
- *The most trivial cone is $\sigma = \{0\}$. In this case $\sigma^\vee = (\mathbb{R}^n)^*$ and $S_\sigma = \mathbb{Z}^n$, hence $X = \mathrm{Spec}(\mathbb{C}[\mathbb{Z}^n] = \mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) = (\mathbb{C}^*)^n$*
- *On the other side of the spectrum, we can take $\sigma = \mathrm{cone}(e_1, \ldots, e_n)$. Then it is clear that this cone is self dual and we obtain $S_\sigma = \mathbb{Z}^n_{\geq 0}$ so that $X = \mathrm{Spec}(\mathbb{Z}^n_{\geq 0}] = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n) = \mathbb{C}^n$*
- *let $\sigma = \mathrm{cone}(2e_1 - e_2, e_2)$. Then the dual cone is generated by the inward pointing normals $\sigma^\vee = \mathrm{cone}(e_1, e_1 + 2e_2)$. Those two generators do not however generate the full semigroup S_σ. Instead we have $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2 = \mathbb{Z}[e_1, e_1 + e_2, e_1 + 2e_2]$. We let $x = \chi^{e_1}, y = \chi^{e_1+e_2}, z = \chi^{e_1+2e_2}$. Then it is clear that $xz - y^2 = 0$, and it 's not hard to in fact chow that this is the only relation so that*

$$\mathbb{V}(xz - y^2) = X_\sigma$$

- *Let $\sigma = \mathrm{cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subset \mathbb{R}^3$. The dual cone is generated by the inward pointing normals $\sigma^\vee = \mathrm{cone}(e_1, e_2, e_3, e_1 + e_2 - e_3)$. We let $x = \chi^{e_1}, y = \chi^{e_2}, z = \chi^{e_3}, w = \chi^{e_1+e_2-e_3}$. Then $xy - zw = 0$ and again this is the only relation so that*

$$\mathbb{V}(xy - zw) = X_\sigma$$

Theorem 2.0.22. *Let X be an affine normal toric variety, then there exists a scrap σ such that $X_\sigma = X$*

The role of normality will be deferred to a next lecture

3 Fans and (non-affine) Toric varieties

We now a rather decent understanding of how scraps relate to affine toric varieties. In this section, we show how one can more generally consider *fans*, built out of scraps (!!) to provide the glueing data required to build more general toric varieties

Definition 3.0.23. a *fan* Σ is a finite collection of scraps such that

- each face of a cone in Σ is a cone in Σ
- the intersection of two cones is a face in each of them

Essentially, we glue scraps together along their faces. To see how a fan defines the glueing data needed to construct abstract varieties, we need the following lemma concerning the intersection of cones:

Theorem 3.0.24. *Let $\tau = \sigma \cap \sigma'$ be a face of to scraps in a fan. Then for each u in the relative interior of $\sigma \cap (-\sigma')^\vee$,*

$$\tau = H_u \cap \sigma = H_u \cap \sigma'$$

Meaning that a face always has an inward pointing normal in both cones. We now glue the different scraps σ in the fan Σ along each face $\sigma \cap \sigma'$ according to the transition morphisms which are the composition of the open embeddings

$$\begin{tikzcd} X_\sigma & & X_{\sigma'} \\ & X_{\sigma\cap\sigma'} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Note that by the above theorem and the fact that going from cone to face is localizing at the character defined by the inward pointing normal, we have that these transition functions are induced by

$$\mathbb{C}[S_\sigma]_{\chi^u} \cong \mathbb{C}[S_{\sigma\cap\sigma'}] \cong \mathbb{C}[S'_\sigma]_{\chi^{-u}}$$

Theorem 3.0.25. *Let Σ be a fan. Then S_Σ is a separated toric variety*

Proof. The above shows that $_\Sigma$ is an abstract variety. To show that that X_Σ contains a torus as an open subset, note that by construction, each cone yields an open affine subset. Since 0 is a cone (as it is a face of each cone), $X_0 \subset X_\Sigma$ yields a torus. Moreover it is an open subset of each X_σ in a compatible way. Since X_0 is dense on each open subset of a finite cover, it is itself dense in X_Σ. To see if the action of X_0 on itself extends to the whole of X_Σ, note that it extends to each X_σ by the affine case, the claim will thus follow since the transition functions are trivially equivariant. It remains to show that $_X\Sigma$ is separated, i.e. whether the diagonal map is closed, this reduces to checking this cone-wise

Theorem 3.0.26. *Let X be a separated toric variety. Then there exists a fan Σ such that $X = X_\Sigma$*

We end the lecture by stating some examples of fans and their associated toric varieties:

Example 3.0.27. *Recall that a possible construction of $\mathbb{P}^2$ is made by glueing together three affine spaces with coordinate rings $\mathbb{C}[x, y]$ with along the open subsets $x \neq 0$ and $y \neq 0$ with transition functions $(x, y) \longrightarrow (\frac{1}{x}, \frac{1}{y})$, yielding the obvious fan.*

Example 3.0.28. *Similarly, the smooth quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ can be defined by the following gluing data:*

Alternatively, we can define the product of two cones $\sigma \times \sigma'$ and extend this to fans $\Sigma \times \Sigma'$ by taking all possible products of cones to prove that $X_\Sigma \times X_{\Sigma'} = X_\Sigma \times X_{\Sigma'}$

Example 3.0.29. *Let 's consider the weighted projective space $X = \mathbb{P}(1, 1, 2)$. We first describe it by gluing affine varieties. In fact the different weightings imply that X is obtained by gluing two affine spaces and a copy of the A_1-singularity:*

4 Recovering the fan of a Toric Variety

The proof of the fact that each toric vareity is built out of a fan is a little too involved, and is an interpretation of the cone-orbit theorem. The subject of next week's talk

References